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BOUNDS ON THERMAL STRESSES IN COMPOSITE BEAMS OF ARBITRARY CROSS-SECTION†

R. B. TESTA and B. BENNETT

Columbia University, New York, New York

and

B. A. BOLEY

Cornell University, Ithaca, New York

BOUNDS on the thermoelastic stresses in homogeneous beams of arbitrary cross-section were derived in [1]. Based on the analysis in [2], which showed the applicability of elementary beam theory to fiber reinforced materials, the bounds were extended to composite beams [3]. Besides their use in design and in estimating errors resulting from uncertainties in temperatures, the bounds have found application in the calculation of lower bounds on the effective torsional rigidity of heated beams [4] and in estimating the error from the use of elementary beam theory in lieu of an exact thermoelastic formulation [5].

The bounds derived in [3] for a composite beam of arbitrary cross-section are applicable only when the following three conditions on the selection of axes are satisfied:

$$\int_A y\,\mathrm{d}A = \int_A z\,\mathrm{d}A = 0 \tag{1}$$

$$\int_A Ey\,\mathrm{d}A = \int_A Ez\,\mathrm{d}A = 0 \tag{2}$$

$$\int_A \alpha ETz\,\mathrm{d}A \int_A Ey^2\,\mathrm{d}A = \int_A \alpha ETy\,\mathrm{d}A \int_A \alpha Eyz\,\mathrm{d}A. \tag{3}$$

In general it is not possible to satisfy all three conditions because they are not independent. However, in one case of practical interest, where fine reinforcing fibers are uniformly distributed throughout the cross section, the bounds in [3] can be used; in that case, equations (1) and (2) are both satisfied by suitably locating the origin ($y = z = 0$), and equation (3) is satisfied by a proper orientation of the axes. Complete generality can be attained by omitting one of the constraints in deriving the bounds. Conditions (1) are omitted in the present work since this seems to lead to the most convenient form for the results.

Let the origin of coordinates be located according to equation (2) and let the orientation of the axes y and z be specified by equation (3). In terms of the dimensionless variables

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defined in [3] and shown in Fig. 1, the expression for the axial stress at any point in a cross-section remains as in [3]:

$$\sigma(\eta, \zeta) = -\tau(\eta, \zeta) + \frac{r(\eta, \zeta)}{A'} \int_{-\eta_1}^{1} \int_{-\zeta_1}^{\zeta_2} \tau(\eta', \zeta') \left[1 + \frac{\eta\eta'}{\rho'^2} \right] \mathrm{d}\eta' \, \mathrm{d}\zeta' . \tag{4}$$

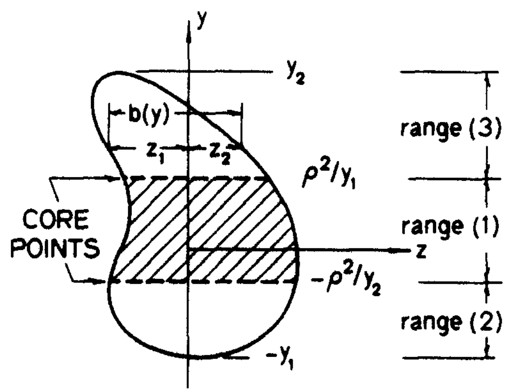

FIG. 1. Cross section and variables: $\eta = y/y_2$; $\zeta = z/y_2$; $\beta(\eta) = b(y)/y_2$; $\rho^2 = \overline{EI}/\overline{AE}$; $\rho'^2 = \rho^2/y_2^2$; $r(\eta, \zeta) = r_i = E(\eta, \zeta)/\bar{E}$; $\overline{EI} = \int_A Ey^2 \, \mathrm{d}A$; $\overline{AE} = \int_A E \, \mathrm{d}A$.

If the extreme values of the temperature variable in a cross-section are known to be

$$-\tau_m \leq \tau(\eta, \zeta) = \alpha ET \leq \tau_M \tag{5}$$

it is desired to find bounds on the stress at any point (η, ζ) such that

$$\sigma_m(\eta, \zeta) \leq \sigma(\eta, \zeta) \leq \sigma_M(\eta, \zeta). \tag{6}$$

In addition, the extreme values of the bounds are sought

$$\begin{aligned} \sum_m &\leq \sigma_m(\eta, \zeta) \\ \sum_M &\geq \sigma_M(\eta, \zeta) \end{aligned} \tag{7}$$

for each of the materials in the composite, to give bounds on the maximum stress acting on each material in the section.

The bounds in equation (6) are obtained as in [3] by considering locations η in the three ranges shown in Fig. 1:

Range 1

$-\rho'^2 < \eta < \rho'^2/\eta_1$:

$$\begin{aligned} \sigma_M(\eta, \zeta) &= \tau_m + r_i \tau_M \left(1 + \frac{Q'}{A'} \frac{\eta}{\rho'^2} \right) \\ \sigma_m(\eta, \zeta) &= -\tau_M - r_i \tau_m \left(1 + \frac{Q'}{A'} \frac{\eta}{\rho'^2} \right). \end{aligned} \tag{8}$$

The quantity

$$Q' = \int_{-\eta_1}^{1} \eta' \beta(\eta')\, d\eta' \tag{9}$$

is the dimensionless first moment of area of the section about the origin specified by equation (2). When equation (1) is also satisfied, as was the case in [3], Q' vanishes and equations (8) together with all subsequent results, reduce to those in [3].

Range 2

$-\eta_1 \le \eta \le -\rho'^2$:

$$\sigma_M(\eta, \zeta) = \frac{r_i}{A'}(\tau_m + \tau_M)\left\{A' - A(\eta) + \frac{\eta}{\rho'^2}[Q' - Q(\eta)]\right\} + \tau_m\left[1 - r_i\left(1 + \frac{Q'}{A'}\frac{\eta}{\rho'^2}\right)\right] \tag{10}$$

$$\sigma_m(\eta, \zeta) = -\frac{r_i}{A'}(\tau_m + \tau_M)\left\{A' - A(\eta) + \frac{\eta}{\rho'^2}[Q' - Q(\eta)]\right\} - \tau_M\left[1 - r_i\left(1 + \frac{Q'}{A'}\frac{\eta}{\rho'^2}\right)\right]$$

where

$$Q(\eta) = \int_{-\rho'^2/\eta}^{1} \eta' \beta(\eta')\, d\eta' = Q' - \int_{-\eta_1}^{-\rho'^2/\eta} \eta' \beta(\eta')\, d\eta' \tag{11}$$

$$A(\eta) = \int_{-\rho'^2/\eta}^{1} \beta(\eta')\, d\eta' = A' - \int_{-\eta_1}^{-\rho'^2/\eta} \beta(\eta')\, d\eta'$$

Range 3

$\rho'^2/\eta_1 \le \eta \le 1$:

$$\sigma_M(\eta, \zeta) = \frac{r_i}{A'}(\tau_m + \tau_M)\left[A(\eta) + \frac{\eta}{\rho'^2}Q(\eta)\right] + \tau_m\left[1 - r_i\left(1 + \frac{\eta}{\rho'^2}\frac{Q'}{A'}\right)\right] \tag{12}$$

$$\sigma_m(\eta, \zeta) = \frac{-r_i}{A'}(\tau_m + \tau_M)\left[A(\eta) + \frac{\eta}{\rho'^2}Q(\eta)\right] - \tau_M\left[1 - r_i\left(1 + \frac{\eta}{\rho'^2}\frac{Q'}{A'}\right)\right].$$

In [3], the maximum value of a bound, equation (7), for any one material (r_i) could be determined by evaluation of the bound at the largest value of η occupied by that material, because the bounds were non-decreasing from $\eta = 0$ to the extreme fibers. In the present case, examination of equations (8)–(12) shows that the bounds do not necessarily reach a minimum at $\eta = 0$ and a general statement on the maximum cannot be made. It will be necessary to evaluate σ_m and σ_M for every location η occupied by any one material r_i; the extremes of these values will be $\sum_m$ and $\sum_M$ for that material.

The upper bound σ_M is illustrated in Figs. 2 and 3, for a rectangular cross section in which equation (3) specifies axes oriented parallel to the edges. The special case $\overline{EI} = \overline{EI}$ is used in Fig. 2 to observe differences in the bounds from those in [3] as the distribution of materials in the section causes the origin of coordinates to shift away from the centroid. Figure 3 shows the dependence of the bound on the location of the core points. A dependence on a new parameter

$$R = \frac{\tau_M}{\tau_M + \tau_m}$$

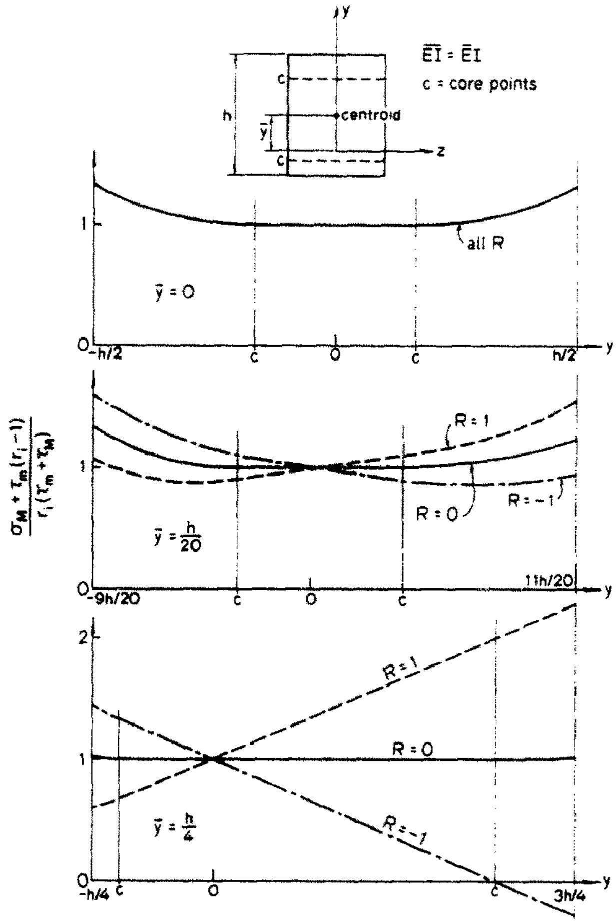

FIG. 2.

which does not appear in [3] when $Q' = 0$ is also shown in the figures for representative values of R.

Although bounds on the maximum stress for each material cannot be determined simply by evaluating the bounds at the largest $|\eta|$ occupied by the material, as was the case in [3], a conservative bound can still be found as in [3] by noting that σ_M and σ_m reach a maximum (in magnitude) at one of the extreme fibers. For σ_M, this is observed in Figs. 2 and 3, and it can be deduced in general from the following equation and inequalities on the function $\sigma_M(\eta, \zeta)$:

$$\frac{\partial \sigma_M}{\partial \eta} = r_i \tau_M \frac{Q'}{\rho'^2 A'} = \text{const.}, \qquad -\rho'^2 < \eta < \frac{\rho'^2}{\eta_1}$$

$$\frac{\partial^2 \sigma_M}{\partial \eta^2} = -r_i(\tau_m + \tau_M)\frac{\beta(\rho'^2/\eta)}{A'}\frac{\rho'^2}{\eta^3} \geq 0, \qquad -\eta_1 \leq \eta \leq -\rho'^2 \tag{13}$$

$$\frac{\partial^2 \sigma_M}{\partial \eta^2} = r_i(\tau_m + \tau_M)\frac{\beta(-\rho'^2/\eta)}{A'}\frac{\rho'^2}{\eta^3} \geq 0, \qquad \frac{\rho'^2}{\eta_1} \leq \eta \leq 1.$$

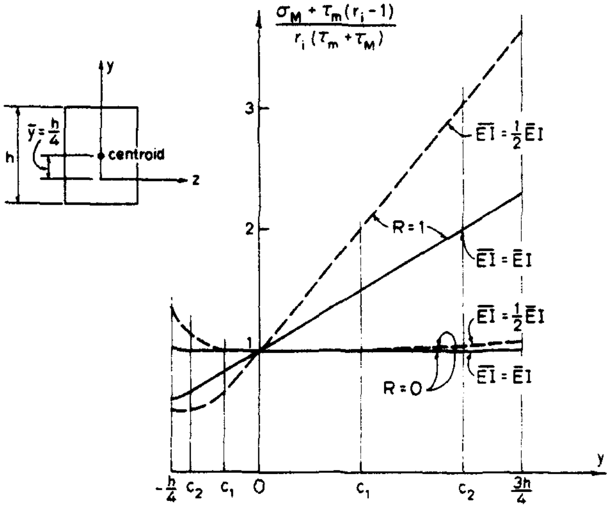

FIG. 3.

Conservative bounds are therefore given by evaluating equations (10) and (12) at the extreme fibers even though a material r_i may not occupy that location:

$$\sum_M = \max[\sigma_M(-y_1, z), \sigma_M(y_2, z)]$$
$$\sum_m = \min[\sigma_m(-y_1, z), \sigma_m(y_2, z)] \tag{14}$$

which can be written in a form similar to that in [3],

$$\sum_M - \tau_m(1-r_i) = \max\left\{r_i(\tau_m+\tau_M)\frac{A_i}{A} + r_i\left[\tau_M + \tau_m\left(1-\frac{Q}{Q_i}\right)\right]\frac{y_iQ_i}{\overline{EI}/\bar{E}}\right\} \tag{15}$$
$$\sum_m + \tau_M(1-r_i) = \min\left\{-r_i(\tau_m+\tau_M)\frac{A_i}{A} - r_i\left[\tau_m + \tau_M\left(1-\frac{Q}{Q_i}\right)\right]\frac{y_iQ_i}{\overline{EI}/\bar{E}}\right\}$$

where A_i, y_i, Q_i take on the values $A_1, -y_1, Q_1$ or A_2, y_2, Q_2. The A_i are the areas from a core point to an extreme fiber as defined in [3] and the Q_i are defined by

$$Q_1 = [Q' - Q(-\eta_1)]y_2^3$$
$$Q_2 = Q(\eta = 1)y_2^3. \tag{16}$$

In [1], a beam column analogy was described to evaluate the expressions corresponding to equations (15) and in [3] a similar analogy was presented to determine the conservative bounds in equation (15) for the special case $\overline{EI} = \bar{E}I$. Extension of the analogy to cover the present case is of questionable value, however, because of the additional terms present when $Q' \neq 0$.

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